Curiosities Regarding the Babylonian Number System

Sherwin Doroudi

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By 2000 BCE the Babylonians were already making significant progress in the areas of mathematics and astronomy and had produced many elaborate mathematical tables on clay tablets. Their *sexagecimal* (base-60) number system was originally inherited from the Sumerian number system dating back to 3500 BCE.[1] However, what made the Babylonian number system superior to its predecessors, such as the Egyptian number system, was the fact that it was a *positional* system not unlike the decimal system we use today. This innovation allowed for the representation of numbers of virtually any size by reusing the same set of symbols, while also making it easier to carry out arithmetic operations on larger numbers. Its superiority was even clear to later Greek astronomers, who would use the sexagecimal number system as opposed to their own native Attic system (the direct predecessor to Roman numerals) when making calculations with large numbers.[2] Most other number systems throughout history have made use of a much smaller base, such as five (*quinary* systems), ten (*decimal* systems), or in the case of the Mayans, twenty (*vigesimal* systems), and such choices for these bases are all clearly related to the fingers on the hands. The choice (though strictly speaking, it's highly unlikely that any number system was directly "chosen") of sixty is not immediately apparent, and at first, it may even seem that sixty is a somewhat large and unwieldy number. Another curious fact regarding the ancient number system of the Babylonians is that early records do not show examples of a "zero" or null place holder, which is an integral part of our own positional number system. This paper will investigate, in brief, these two curiosities regarding the Babylonian number system.

Although historians and mathematicians alike have long sought to discover why the Babylonians used sixty as the base for their number systems, to this day there is no conclusive answer to this elusive question. One of the most common theories on the origin of the base-60 system, which has been suggested by many authors is that the base was a result of the merger between different number systems.[3] As a hypothetical example, imagine a tribe that used a *vigesimal* (base-5) system, and would count using the fingers on one hand, and another that used a *duodecimal* (base-12) system and would count using the twelve finger segments on one hand (as separated by the joints, and ignoring the thumb). Indeed, using this system one could count up to sixty using both hands, by using fingers in one hand to represent twelves, and using them to point to a segment in the other hand representing the units. If these tribes began to engage in regular trade, a common system would be beneficial to both. It should be noted however, that the Babylonian system of representing numbers through cuneiform writing uses symbols for units and tens, putting these symbols together to create fifty-nine unique aggregate symbols, which are positioned to represent any natural number. This would perhaps indicate that a merger between a base-6 and a base-10 system may have been more likely than one between base-5 and base-12 systems. Whatever the case, the base-60 system was not one that would disappear quickly, and in fact, we still continue to use remnants of this ancient system in measuring time and angles.

Some theories regarding the origin of the base-60 system, are concerned with the versatility of the number sixty, as it has so many factors; in particular sixty is the smallest positive integer having as factors all of the integers between one and six, inclusive. These factors are particularly useful for division by common numbers, and in this respect this number system puts even our current decimal system to shame when it comes to the representing fractions using floating points.[4] Consider the following true example; historians were long puzzled by the frequency with which certain (seemingly) large numbers were appearing on ancient Babylonian tablets. One such number was $[44, 26, 40]_{60} = 44 \times 60^2 + 26 \times 60 + 40 = 160\ 000$, for which many multiplication tables were found. When this number also appeared on tables of reciprocals, in particular, as the reciprocal of $[1, 21]_{60} = 1 \times 60 + 21 = 81$, it became apparent that this use of $[44, 26, 40]_{60}$ was actually $44 \times 60^{-2} + 26 \times 60^{-3} + 40 \times 60^{-4} = \frac{1}{81}$, the reciprocal of $[1, 21]_{60}$, as expected.[5] This discovery serves to show that the Babylonians could represent more rational numbers as finitely terminating floating points, than they would have been able to using a smaller base, thus making division computations simpler (e.g. dividing by $[1, 21]_{60}$ became analogous to multiplying by $[44, 26, 40]_{60}$).[4] It is worth noting that such "sixty is nice theories" and the "merger theory" are not necessarily mutually exclusive, making both quite plausible.

The other curious fact regarding the Babylonian number system is its lack of a symbol for "zero," (for at least the first 1000 years in which the system was in use) and moreover its lack of a radix point (known as a decimal point in decimal systems and denoted by the full stop symbol "." in some countries and the comma symbol "," in others). Using a symbol for zero we could differentiate $[1, 0, 15]_{60} = 3615$ from $[1, 15]_{60} = 75$, while without such a symbol we must resort to context to determine the precise number (arithmetic context, where explicit calculations are shown, can be particularly helpful). The lack of a radix point is no less ambiguous, as we know the Babylonians did indeed use floating point numbers. The confusion mentioned above regarding the use of the number $[44, 26, 40]_{60}$ to represent "$[\textit{radix point}, 0, 44, 26, 40]_{60}$," arises precisely due to the lack of both a zero and a radix point. Interestingly, however, there is no evidence that the Babylonians ever had any problems due to what we now perceive as ambiguities. When placeholders for zero finally did emerge (the concept of zero as an integer did not make its debut until some time later in India), there was no standard for the new symbol, so different tablets used one of several symbols for zero. Moreover, no examples have been found that use this symbol at the end of a number (i.e. the zero is never used in the units place), which also introduces ambiguities akin to those arisen from the lack of a radix point, though in most cases such ambiguities can once again be resolved using the given context.[2] Eventually another ambiguity regarding the Babylonian zero began to surface. Symbols for zero were also being used to represent numbers such as 1201 by $[20, 0, 1]_{60'}$ (where the prime denotes the use of the alternate zero), whereas 1201 would normally be represented by $[20, 1]_{60}$ and $[20, 0, 1]_{60}$ would normally represent $20 \times 60^2 + 0 \times 60 + 1 = 72\ 001$. It was later discovered that tablets would use either one type of zero or the other, never both, and that it was only used to separate factors of ten by numbers less than ten. That is, this type of zero was used to prevent "ten" symbols and "unit" symbols from being grouped together and treated as being in the same sexagecimal position, rather than to show they were two positions apart. Once again, such ambiguities can only be resolved by contextual information.[6] It is interesting to note, that although the Babylonians had come up with two different notions of zero as a placeholder, each to resolve a different type of ambiguity, for some time they created even more ambiguities for modern-day historians, perhaps because they were not always differentiated. Were these two notions of zero used together with two different, standardized symbols, and if a radix point symbol had been created, the Babylonians would have been a few step closer to our modern number system.

Investigating curiosities such as the "choice" of a base and the use of zeroes and radix points regarding any number system, serve to show how far along we've come to using our sophisticated, yet seemingly simple number system, while at the same time showing that even ancient systems have some advantages over ours. Continual investigation of such curiosities, will help us move even further along in representing information with greater clarity, efficiency, and practicality.

References

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